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Application of the SUSY semiclassical quantisation rule to non-solvable potentials

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Abstract. General features of the SUSY semiclassical quantisation rule including higher-order corrections are discussed in relation to non-solvable potentials. The pairing between the levels of the SUSY partner Hamiltonians is shown to hold in every order of the approximation up to the fifth. This result appears to be true in all orders. A class of non-solvable superpotentials $\phi = x^{2N+1}/2N+1$ is studied in some detail and numerical results for the case $N=1$ are presented.

1. Introduction

Motivated by considerations of supersymmetry, Comtet *et al* (1985) have proposed a modified JWKB quantisation rule which is applicable to one-dimensional Hamiltonians containing a potential of the form $V(x) = \phi^2 + \hbar\, \mathrm{d}\phi/\mathrm{d}x$. This SJWKB rule, derived in the lowest order of a semiclassical treatment of the bound-state problem, was found to yield the exact spectrum for known solvable potentials, including those for which the standard (lowest-order) JWKB quantisation condition is not exact. This remarkable property of the SJWKB rule was recently shown by us (Raghunathan *et al* 1987) to be due to the vanishing of all higher-order corrections for all solvable potentials, a class which includes the so-called shape-invariant potentials (Dutt *et al* 1986).

In contrast, the higher-order corrections for non-solvable potentials are non-zero and the lowest-order SJWKB rule is not exact. It is therefore of interest to examine how well the SJWKB quantisation rule works in such cases. To this end, the SJWKB analysis must be extended to include the higher orders of the approximation. The importance of higher-order corrections in the traditional JWKB formalism is now well documented (see, for instance, Bender *et al* 1977, Seetharaman and Vasan 1984, 1986).

Our aim in the present work is to study the higher orders in the SJWKB analysis, with the view to determine (i) the general features, if any, of the SJWKB scheme in higher orders, (ii) whether the relation $E_{+,K} = E_{-,K+1}$ between the exact eigenvalues of the partner Hamiltonians $H_\pm = p^2 + (\phi^2 \pm \hbar\, \mathrm{d}\phi/\mathrm{d}x)$ holds also for the SJWKB energy values in every order of the approximation, and (iii) the extent to which the analysis parallels the conventional JWKB approach. We follow the method of Dunham (1932) which enables us to extend the SJWKB rule to all orders in an elegant fashion. We analyse in some detail a class of superpotentials given by $\phi = x^{2N+1}/2N+1$ and have carried out calculations up to the twelfth order.

This paper is organised as follows. In the next section we show how the SJWKB rule can be extended to all orders and discuss some general features of the resulting

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analysis. In § 3 we apply the formalism to the case of the superpotential $\phi = x^{2N+1}/2N+1$. Results are presented and discussed in the last section.

2. Higher-order corrections to the SJWKB rule

In SUSYQM one considers partner Hamiltonians

$$H_\varepsilon = -\hbar^2 \frac{d^2}{dx^2} + \phi^2(x) + \varepsilon\hbar \frac{d\phi}{dx} \qquad \varepsilon = \pm 1. \tag{2.1}$$

The SJWKB rule of Comtet *et al* (1985) for determining the eigenvalues of H_ε is

$$\int_a^b (E_\varepsilon - \phi^2)^{1/2}\, dx = \pi\hbar(K + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) \qquad K = 0, 1, 2, \ldots \tag{2.2}$$

where a, b are turning points defined by $\phi^2(a) = \phi^2(b) = E_\varepsilon$. To derive (2.2), one proceeds in the usual manner: the wavefunction is approximated in the form $\exp[i(S_0 + \hbar S_1)/\hbar]$ and the connection formulae are invoked to impose continuity.

Long ago Dunham (1932) showed that the standard (lowest-order) JWKB quantisation rule could be generalised to include all higher-order corrections. In the SUSY context, when ϕ^2 has a single minimum, Dunham's analysis can be readily adapted to yield the following quantisation rule (with $\hbar = 1$):

$$\oint \sum_{n=0}^{\infty} (-i)^n y_n\, dx = 2\pi K \qquad K = 0, 1, 2, \ldots. \tag{2.3}$$

The functions $y_n(x)$ are to be determined by solving the recurrence relation

$$y_{n+1} = -\frac{1}{2y_0}\left(\frac{dy_n}{dx} + \sum_{m=1}^{n} y_m y_{n+1-m}\right) \qquad n = 1, 2, \ldots \tag{2.4}$$

with

$$y_0^2 = E_\varepsilon - \phi^2(x) \qquad y_1 = -\frac{1}{2y_0}\left(\frac{dy_0}{dx} + i\varepsilon\frac{d\phi}{dx}\right). \tag{2.5}$$

The integration in (2.3) is over a closed contour in the complex x plane enclosing a branch cut along the real axis joining the two real branch points of y_0. As the direction of integration depends on the branch of y_0, we take for definiteness y_0 to be that branch which is negative real on the upper lip of the cut. The contour is then to be traversed in the counterclockwise sense. We may note that the recurrence relation occurring in Dunham's JWKB formalism is the same as (2.4) but the expressions for y_0 and y_1 differ from those given in (2.5). The quantisation formula (2.3) is exact and forms the basis of all higher-order calculations considered in this work. It is not difficult to verify that, if the $n = 0$ and $n = 1$ terms alone are retained in (2.3), the SJWKB rule (2.2) is obtained. Thus (2.3) is the generalisation to all orders of (2.2).

It is evident from (2.2) that $E_{+,K} = E_{-,K+1}$. In other words, the lowest-order SJWKB rule reproduces the pairwise degeneracy that is known to occur between the exact levels of the SUSY partners $H_\pm$ (Witten 1981, Sukumar 1985). We now investigate whether the inclusion of higher-order corrections to (2.2) preserves this symmetry. From the recurrence relation (2.4) explicit expressions for the y_n in terms of y_0 and ϕ can be obtained. As $\varepsilon^2 = 1$, every y_n can be written as

$$y_n = a_n + \varepsilon b_n \tag{2.6}$$

where a_n, b_n are independent of ε. It is clear that if the b_n do not contribute to (2.3), the pairing of levels will be preserved.

Substituting (2.6) in (2.4) we get the following recurrence relations for the a_n and b_n:

$$\frac{\mathrm{d}a_n}{\mathrm{d}x} = -\sum_{m=0}^{n+1} (a_m a_{n+1-m} + b_m b_{n+1-m}) \qquad n = 0, 1, 2, \ldots \tag{2.7}$$

$$\frac{\mathrm{d}b_n}{\mathrm{d}x} = -\sum_{m=0}^{n+1} (a_m b_{n+1-m} + b_m a_{n+1-m}) - \mathrm{i}\frac{\mathrm{d}\phi}{\mathrm{d}x}\,\delta_{n,0} \tag{2.8}$$

with $a_0 = y_0$, $b_0 = 0$. To proceed further, we adapt a method due to Fröman (1966). We define

$$A_E = \sum_{n=0}^{\infty} \lambda^{2n} a_{2n} \qquad A_0 = \sum_{n=0}^{\infty} \lambda^{2n+1} a_{2n+1}$$
$$B_E = \sum_{n=0}^{\infty} \lambda^{2n} b_{2n} \qquad B_0 = \sum_{n=0}^{\infty} \lambda^{2n+1} b_{2n+1} \tag{2.9}$$

where λ is an arbitrary parameter. By virtue of (2.7) and (2.8), we get the following relations among the A and B:

$$A'_E = -(2/\lambda)(A_0 A_E + B_0 B_E) \tag{2.10}$$

$$A'_0 = -(1/\lambda)(A_E^2 + B_E^2 + A_0^2 + B_0^2 - a_0^2) \tag{2.11}$$

$$B'_E = -(2/\lambda)(A_0 B_E + A_E B_0) - \mathrm{i}\phi' \tag{2.12}$$

$$B'_0 = -(2/\lambda)(A_E B_E + A_0 B_0). \tag{2.13}$$

Here the primes on the LHS denote differentiation with respect to x. In obtaining the above we have made use of $b_0 = 0$. From (2.10) and (2.13), it follows that

$$A'_E \pm B'_0 = -(2/\lambda)(A_E \pm B_0)(A_0 \pm B_E).$$

Hence

$$A_0 \pm B_E = -\frac{\lambda}{2}\frac{\mathrm{d}}{\mathrm{d}x}\ln(A_E \pm B_0). \tag{2.14}$$

Equation (2.14) shows that

$$B_E = -\frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(A_E + B_0) - \ln(A_E - B_0)]. \tag{2.15}$$

Expanding both sides in powers of λ, it can be seen that every b_{2n} is a derivative. Consequently, $\oint b_{2n}\,\mathrm{d}x$ vanishes. It then follows that the even-order corrections in (2.3) taken by themselves preserve the level pairing. As regards b_{2n+1} ($n = 1, 2, \ldots$), we first obtain the following explicit expressions for b_3 and b_5 from the recurrence relations (2.7) and (2.8):

$$b_3 = -(\mathrm{i}/16 y_0^7)[25\phi^2\phi'^3 + y_0^2(5\phi'^3 + 16\phi\phi'\phi'') + 2y_0^4\phi''']$$

$$b_5 = (\mathrm{i}/256 y_0^{13})[12\,155\phi^4\phi'^5 + 2y_0^2(3453\phi^2\phi'^5 + 6188\phi^3\phi'^3\phi'')$$
$$+ y_0^4(399\phi'^5 + 4344\phi\phi'^3\phi'' + 2300\phi^2\phi'\phi''^2 + 1668\phi^2\phi'^2\phi''')$$
$$+ 4y_0^6(89\phi'\phi''^2 + 63\phi'^2\phi''' + 66\phi\phi''\phi''' + 38\phi\phi'\phi'''') + 8y_0^8\phi''''']. $$

It is a simple matter to verify that these are expressible as derivatives of single-valued functions:

$$b_3 = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{5\phi\phi'^2}{16y_0^5}+\frac{\phi''}{8y_0^3}\right)$$

$$b_5 = \frac{\mathrm{i}}{256}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{y_0^{11}}[1105\phi^3\phi'^4 + y_0^2(399\phi\phi'^4 + 884\phi^2\phi'^2\phi'') + 4y_0^4(35\phi'^2\phi'' + 19\phi\phi''^2 + 28\phi\phi'\phi''') + 8y_0^6\phi'''']\right).$$

We believe this to be true of all b_{2n+1}. We have been unable to find a general proof of this conjecture so far, but considerable evidence in its favour comes from our study of a class of superpotentials $\phi = x^{2N+1}$ (discussed below). If b_{2n+1} is a derivative as conjectured, then the level pairing will hold in every order.

In view of the above discussion, it is clear that the corrections to the lowest-order result in (2.3) come only from the a_n. Of these, the contribution from the odd terms, a_{2n+1}, can be determined easily because A_0 is given by

$$A_0 = -\frac{\lambda}{4}\frac{\mathrm{d}}{\mathrm{d}x}[\ln(A_E + B_0) + \ln(A_E - B_0)] \qquad (2.16)$$

which follows from (2.10) and (2.13). Expanding the RHS of (2.16) in powers of λ and matching terms, it is clear that a_{2n+1} is a derivative. Apart from a_1, which is a logarithmic derivative, the other a_{2n+1} do not contribute to the quantisation rule (2.3). Therefore, the quantisation rule can be written as

$$-\mathrm{i}\oint (a_1 + \varepsilon b_1)\,\mathrm{d}x + \oint \sum_{n=0}^{\infty}(-1)^n a_{2n}\,\mathrm{d}x = 2\pi K$$

i.e.

$$\oint \sum_{n=0}^{\infty}(-1)^n a_{2n}\,\mathrm{d}x = 2\pi(K + \tfrac{1}{2} + \tfrac{1}{2}\varepsilon) \qquad K = 0, 1, 2, \ldots . \qquad (2.17)$$

This quantisation formula is very similar to what one has in the standard JWKB treatment (Bender *et al* 1977). However, we may note that the a_{2n} are not the same as those in the JWKB analysis because of the b_n. The extra $\frac{1}{2}\varepsilon$ on the RHS reflects the cancellation of the zero-point energy of the ground state in SUSYQM. The above formula forms the basis for our calculations of energies in the SJWKB scheme.

3. Application to the case $\phi = x^{2N+1}/2N+1$

We now consider the class of superpotentials $\phi = x^{2N+1}/2N+1$ which gives $V_\varepsilon(x) = x^{4N+2}/(2N+1)^2 + \varepsilon x^{2N}$. Both V_+ and V_- are confining potentials. Unlike V_+, V_- is a double-well potential for where there exists a normalisable ground state with $E_- = 0$ and wavefunction $\sim\exp[-x^{2N+2}/(2N+1)(2N+2)]$. As V_+ has no normalisable state with $E_+ = 0$, this is a case of unbroken SUSY.

Using the recurrence relation (2.4), we find that y_n for any n can be expressed in the form

$$y_n = (\mathrm{i}\varepsilon)^n \sum_{k=0}^{[n\alpha]} C_{n,k}(\mathrm{i}\varepsilon x)^{(4N+1)n-(2N+1)k} y_0^{k+1-3n} \qquad \alpha = \frac{4N+1}{2N+1} \qquad (3.1)$$

where $C_{n,k}$ are constants, and $[p]$ stands for the integer part of p. From (3.1) and (2.4) the following recurrence relation for the coefficients $C_{n,k}$ is obtained:

$$-2C_{n+1,m}=[(4N+1)n-(2N+1)(m-2)]C_{n,m-2}+\frac{(-3n+m+1)}{2N+1}C_{n,m}$$
$$+\sum_{j=1}^{n}\sum_{k=0}^{[j\alpha]}C_{j,k}C_{n+1-j,m-k} \qquad (3.2)$$

The values of the first few sets of the $C_{n,k}$ computed from (3.2) using the initial values $C_{0,0}=1$, $C_{1,0}=-1/(4N+2)$, and $C_{1,1}=\frac{1}{2}$ are given in table 1.

To evaluate the integral of y_n in (2.3), we need to compute integrals of the two types

$$\oint x^{2m}y_0^{-(2n+1)}\,dx \qquad \oint x^{2m+1}y_0^{-2n}\,dx$$

for non-negative integral values of m and n. The evaluation of such integrals is outlined in appendix 1, and we have the results:

$$\oint\frac{x^{2m}}{y_0^{2n+1}}\,dx=\frac{(-2)^n(4n+2)}{(2n+1)!!}(2N+1)^{2(m-N)/(2N+1)}$$
$$\times B(\tfrac{1}{2},(2m+1)/(4N+2))\frac{d^n}{dE^n}E^{(m-N)/(2N+1)} \qquad (3.3)$$

$$\oint\frac{x^{2m+1}}{y_0^{2n}}\,dx=\frac{2\pi i(-1)^n}{(n-1)!}(2N+1)^{(2m-2N+1)/(2N+1)}\frac{d^{n-1}}{dE^{n-1}}E^{(m-2N)/(2N+1)} \qquad (3.4)$$

In the above $B(s,t)$ is the beta function defined by $B(s,t)=\Gamma(s)\Gamma(t)/\Gamma(s+t)$.

Using the above formulae we can write down the quantisation rule to any order. We have carried out this calculation up to twelfth order. After identifying the a_n and b_n of (2.6) in (3.1), we find that b_n is a derivative and hence $\oint b_n\,dx$ vanishes for every $n>1$, whereas $\oint b_1\,dx=i\pi$. This implies that the pairing of energy levels of the SUSY partners $H_\pm$ is preserved in every order of the SJWKB approximation considered. Not surprisingly, we also find that $\oint a_{2n+1}\,dx=0$ for every $n>0$. Collecting together the

Table 1. Values of $Q_{nk}=(2N+1)^nC_{n,k}$.

$Q_{00}=1$	$Q_{10}=-1/2$	$Q_{11}=(2N+1)/2$	
$Q_{20}=-5/8$	$Q_{21}=(2N+1)/2$	$Q_{22}=(6N+1)(2N+1)/8$	$Q_{23}=-N(2N+1)^2/2$
$Q_{30}=-15/8$	$Q_{31}=25(2N+1)/16$	$Q_{32}=(56N+10)(2N+1)/16$	
$Q_{33}=-(42N+5)(2N+1)^2/16$	$Q_{34}=-N^2(2N+1)^2$	$Q_{35}=N(2N-1)(2N+1)^3/4$	
$Q_{40}=-1105/128$	$Q_{41}=15(2N+1)/2$	$Q_{42}=(1418N+267)(2N+1)/64$	
$Q_{43}=-(145N+23)(2N+1)^2/8$	$Q_{44}=-(1748N^2+316N+21)(2N+1)^2/128$		
$Q_{45}=(39N^2+2N)(2N+1)^3/4$			
$Q_{46}=(10N^3-5N^2+N)(2N+1)^3/8$	$Q_{47}=-N(2N-1)(N-1)(2N+1)^4/4$		
$Q_{50}=-1695/32$	$Q_{51}=12\,155(2N+1)/256$	$Q_{52}=(5580N+1095)(2N+1)/32$	
$Q_{53}=-(19\,282N+3453)(2N+1)^2/128$	$Q_{54}=-(2696N^2+698N+53)(2N+1)^2/16$		
$Q_{55}=(34\,844N^2+6948N+399)(2N+1)^3/256$	$Q_{56}=(370N^3+9N^2+3N+3)(2N+1)^3/8$		
$Q_{57}=-(1024N^3-182N^2+13N)(2N+1)^4/32$	$Q_{58}=(2N-1)(-6N^3+5N^2-N-1)(2N+1)^4/8$		
$Q_{59}=N(N-1)(2N-1)(2N-3)(2N+1)^5/8$			

contributions from the non-vanishing integrals, the quantisation rule can be expressed as

$$2\pi(K+\tfrac{1}{2}+\tfrac{1}{2}\varepsilon)=\sum_{n=0} A_{2n}E_\varepsilon^{(N+1)(1-2n)/(2N+1)} \qquad K=0,1,2,\ldots \qquad (3.5)$$

where the coefficients A_{2n} are functions of N. The energy values can be computed from this formula. To illustrate let us choose $N=1$. The values of A_{2n} for this case up to $n=6$ are listed below:

$$A_0=\tfrac{1}{2}3^{1/3}B(\tfrac{1}{2},\tfrac{1}{6})=5.254\,080\,527 \qquad A_2=\tfrac{2}{27}3^{2/3}B(\tfrac{1}{2},\tfrac{5}{6})=0.345\,217\,275$$
$$A_4=0 \qquad A_6=-\tfrac{2636}{98415}3^{1/3}B(\tfrac{1}{2},\tfrac{1}{6})=-0.281\,456\,206 \qquad (3.6)$$
$$A_8=-1.046\,313\,017 \qquad A_{10}=0 \qquad A_{12}=101.4.$$

The values of A_0 and A_2 given above coincide with the ones computed by Dutt *et al* (1986) who have applied the SJKWB quantisation rule to the potential $V=\frac{1}{9}x^6+x^2$ taking only the first two terms in (3.5). As noted by them, reasonably good results are obtained with just A_0 and A_2 in (3.5).

4. Results and discussion

The energy values corresponding to the potential $V_+(x)=\frac{1}{9}x^6+x^2$ calculated using (3.5) are presented in table 2. The exact energies shown in the table are the numerical eigenvalues computed by Boya *et al* (1987) (suitably scaled to correspond to our V_+). The last two exact values are from the work of Mathews *et al* (1981a, b). The agreement between the SJWKB values and the exact eigenvalues is quite good. In this connection we wish to point out that the 'exact' values quoted by Dutt *et al* (1986) for the above potential are inaccurate because they have approximated the coupling strength $\frac{1}{9}$ by 0.1. This results in their values being in significant error, the magnitude of the error increasing with the quantum number K of the level. Our results show that corrections to the energy values beyond order $n=2$ are generally quite small. Nevertheless, the asymptotic nature of the SJWKB expansion is clearly reflected in the numerical values for the low-lying levels obtained in various orders. Another indication of this is the manner in which the coefficients A_{2n} in (3.5) vary with n.

Table 2. SJWKB energy values for the potential $V_+=\frac{1}{9}x^6+x^2$.

	E_K				
K	zeroth order	second order	sixth order	eighth order	Exact
0	1.307 75	1.214 206	1.256 751	1.336 743	1.117 451
1	3.698 876	3.634 586	3.636 277	3.637 398	3.636 438
2	6.795 268	6.743 033	6.743 297	6.743 373	6.744 012
3	10.462 00	10.416 84	10.416 91	10.416 92	10.416 92
4	14.621 09	14.580 73	14.580 75	14.580 75	14.580 75
5	19.219 92	19.183 09	19.183 10	19.183 10	19.183 10
10	47.710 38	47.683 29	47.683 29	47.683 29	47.683 29
100	1 327.415	1 327.406	1 327.406	1 327.406	1 327.406
1000	41 416.73	41 416.73	41 416.73	41 416.73	41 416.73

Finally, we make a few general observations regarding the $\hbar$ series expansion in the JWKB and SJWKB approaches. It was pointed out in our earlier paper (Raghunathan *et al* 1987) that for solvable potentials, the lowest-order SJWKB term, which is exact, can be expanded in a power series in $\hbar$ and the series so obtained coincides term by term with the corresponding JWKB series for that potential. For the non-solvable potential that we have studied in the present work, each order of SJWKB corresponds to a definite power of $\hbar$ while each order of JWKB can further be expanded in a power series in $\hbar$. A closer look at the JWKB series reveals that the nth-order term contains all terms of the SJWKB series starting with $\hbar^n$ but with different coefficients. In the light of this, it is not possible to say, up to any given (finite) order, whether SJWKB analysis would be better or worse than conventional JWKB analysis. However, it can be verified that if each JWKB term is expanded in powers of $\hbar$ and all terms of a given power of $\hbar$ collected then the resulting coefficients will be the same as those in the SJWKB series. Some of the above points are illustrated in appendix 2 for the case $\phi = x^3/3$.

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Appendix 1

We evaluate here the integrals in (3.3) and (3.4). Consider

$$I(m, n) = \oint x^{2m} y_0^{-(2n+1)}\, dx \tag{A1.1}$$

where $y_0 = [E - x^{4N+2}/(2N+1)^2]^{1/2}$ and the contour is a closed curve enclosing only a branch cut on the real axis connecting the points $x = \pm[(2N+1)^2 E]^{1/(4N+2)}$. By differentiation under the integral sign, (A1.1) could be rewritten as

$$I(m, n) = \frac{(-2)^n}{(2n-1)!!} \frac{d^n}{dE^n} I(m, 0). \tag{A1.2}$$

By a change of variable $x = t[(2N+1)^2 E]^{1/(4N+2)}$ one can separate out the E dependence of the integral as follows:

$$I(m, n) = \frac{(-2)^n}{(2n-1)!!} (2N+1)^{(2m+1)/(2N+1)} \frac{d^n}{dE^n} E^{(m-N)/(2N+1)} \oint \frac{t^{2m}\, dt}{(1-t^{4N+2})^{1/2}}. \tag{A1.3}$$

Since the integrand in (A1.3) has only integrable singularities at $t = \pm 1$, the contour can be compressed onto the real axis. It then follows that

$$\begin{aligned}\oint \frac{t^{2m}\, dt}{(1-t^{4N+2})^{1/2}} &= 2\int_{-1}^{1} \frac{t^{2m}\, dt}{(1-t^{4N+2})^{1/2}} \\ &= \frac{2}{2N+1} \int_0^1 dy (1-y)^{-1/2} y^{(2m-4N-1)/(4N+2)} \\ &= \frac{2}{2N+1} B(\tfrac{1}{2}, (2m-4N-1)/(4N+2)+1).\end{aligned} \tag{A1.4}$$

Using (A1.4) and (A1.3) we get immediately (3.3). Proceeding similarly, the integral

$$J(m, n) = \oint x^{2m+1} y_0^{-2n} \, dx \tag{A1.5}$$

can be written as

$$J(m, n) = \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{dE^{n-1}} J(m, 1). \tag{A1.6}$$

In this case the only singularities enclosed by the contour are a pair of simple poles at $t = \pm 1$. Applying the residue theorem, it is simple to show that

$$J(m, 1) = -2\pi i(2N+1)^{(2m+1-2N)/(2N+1)} E^{(m-2N)/(2N+1)}. \tag{A1.7}$$

Inserting (A1.7) into (A1.6), we obtain (3.4).

Appendix 2

For a potential $V(x)$ with a single minimum the traditional JWKB quantisation condition can be written as

$$\sum_{n=0}^{\infty} \oint (-i\hbar)^{2n} y_{2n} \, dx = 2\pi(K + \tfrac{1}{2})\hbar \tag{A2.1}$$

where the y_n satisfy the recurrence relation (2.4) with $y_0^2 = E - V(x)$ and $y_1 = -(1/2y_0) \, dx_0/dx$. For

$$V(x) = \tfrac{1}{9}x^6 + \hbar x^2 \tag{A2.2}$$

the lowest-order term in (A2.1) is

$$\oint y_0 \, dx = \oint (E - \tfrac{1}{9}x^6 - \hbar x^2)^{1/2} \, dx.$$

Expanding this as a power series in $\hbar$, one obtains

$$\oint y_0 \, dx = \tfrac{1}{2}B_1 E^{2/3} - \pi\hbar + \tfrac{1}{6}B_5 E^{-2/3}\hbar^2$$
$$+ \tfrac{1}{36}B_1 E^{-4/3}\hbar^3 - \tfrac{5}{216}B_5 E^{-8/3}\hbar^5 - \tfrac{49}{6480}B_1 E^{-10/3}\hbar^6 + \ldots \tag{A2.3}$$

where

$$B_1 = 3^{1/3} B(\tfrac{1}{2}, \tfrac{1}{6}) \qquad B_5 = 3^{2/3} B(\tfrac{1}{2}, \tfrac{5}{6}). \tag{A2.4}$$

Clearly, the lowest-order JWKB integral for the potential (A2.2) contains all powers of $\hbar$. This is true of every $\oint y_{2n} \, dx$. Proceeeding as above we get

$$-\oint y_2 \, dx = -\tfrac{5}{54}B_5 E^{-2/3} - \tfrac{1}{36}B_1 E^{-4/3}\hbar + \tfrac{95}{972}B_5 E^{-8/3}\hbar^3 + \tfrac{7}{144}B_1 E^{-10/3}\hbar^4 + \ldots \tag{A2.5}$$

$$\oint y_4 \, dx = -\tfrac{145}{1944}B_5 E^{-8/3}\hbar - \tfrac{4907}{58\,320}B_1 E^{-10/3}\hbar^2 + \ldots \tag{A2.6}$$

$$-\oint y_6 \, dx = \tfrac{5135}{314\,928}B_1 E^{-10/3} + \ldots . \tag{A2.7}$$

Multiplying by appropriate powers of $\hbar$ and adding, the LHS of (A2.1) correct to order $\hbar^6$ is

$$\tfrac{1}{2}B_1E^{2/3}-\pi\hbar+\tfrac{2}{27}B_5E^{-2/3}\hbar^2-\tfrac{2636}{98\,415}B_1E^{-10/3}\hbar^6. \qquad \text{(A2.8)}$$

Transferring $\pi\hbar$ to the RHS of (A2.1) and comparing with (3.5), it is seen that the above coefficients of different powers of E are the same as those in (3.6). This illustrates the point made in § 4 of the text.

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